

**Math 241**  
**Spring 2018**  
**Exam 2 - Practice**  
**3/12/18**  
**Time Limit: 50 Minutes**

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Name (Print): \_\_\_\_\_ **KEY**

Problem	Points	Score
1	20	
2	20	
3	20	
4	15	
5	20	
6	10	
7	20	
Total:	125	

1. (20 points) Use linearization to approximate  $\sqrt{143}$  and  $\sqrt[3]{124}$ .

$\sqrt{143}$ : let  $f(x) = \sqrt{x}$  and note that  $f'(x) = \frac{1}{2\sqrt{x}}$ .

We "linearize"  $f(x)$  at  $x=144$  because it's close to 143 and  $f(144) = 12$  (it's easy to compute). We have

$$L_{144}(x) = \frac{1}{24}(x - 144) + 12$$

$\uparrow$                              $\uparrow$   
 $f'(144)$                          $f(144)$

This gives  $L_{144}(143) = 12 - \frac{1}{24} \approx \sqrt{143}$ .

$\sqrt[3]{124}$ : let  $f(x) = \sqrt[3]{x}$ .  $f'(x) = \frac{1}{3 \cdot (\sqrt[3]{x})^2}$

$$f(125) = 5 \quad \text{and} \quad \cancel{f'(125)} = \frac{1}{75}$$

$$L_{125}(x) = \frac{1}{75}(x - 125) + 5$$

$$\text{Now, } \sqrt[3]{124} \approx L_{125}(124) = 5 - \frac{1}{75}$$

2. (a) (10 points) State Rolle's Theorem and the Mean Value Theorem.

Rolle's Thm: Given a continuous  $f$  on  $[a,b]$ , which is differentiable on  $(a,b)$ , then, If  $f(a)=f(b)$ , then  $f'(c)=0$  for some  $c$  in  $(a,b)$ .

MVT:  $f$  continuous on  $[a,b]$ , differentiable on  $(a,b)$ , then there is some  $c \in (a,b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) (10 points) Use the IVT and Rolle's Theorem (or the Mean Value Theorem) to show that  $2x - \sqrt{2} = \cos^2(x)$  has one, and only one solution.

Let  $f(x) = \cos^2(x) - 2x + \sqrt{2}$ . It's continuous.

$$f(0) = 1 + \sqrt{2} \quad \text{and} \quad f\left(\frac{3\pi}{2}\right) = -3\pi + \sqrt{2} \quad (\text{a negative } \#).$$

The IVT gives an  $\alpha$  in  $[0, \frac{3\pi}{2}]$  such that  $f(\alpha) = 0$ .

This  $\alpha$  is a solution to that stupid equation.

If  $f$  had more than 1 zero (i.e. the equation has more than one zero), then, because  $f$  is differentiable on all of  $\mathbb{R}$ , Rolle's theorem would give a  $c$  such that  $f'(c)=0$ .

$$\begin{aligned} \text{BUT, } f'(x) &= 2\cos(x)(-\sin(x)) - 2 \\ &= -2\sin(2x) - 2 < 0 \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

so,  $f$  can only have 1 zero, i.e. the equation can only have one solution.

3. (20 points) Suppose that a particle has an acceleration function  $a(t) = 12t^2 + 2t$ . If the velocity function,  $v(t)$  has the property that  $v(1) = 1$ , and the position function,  $p(t)$ , has the property that  $p(1) = 0$ , find explicit formulas for  $v(t)$  and  $p(t)$ .

$$V(t) = 4t^3 + t^2 + C_1$$

$$1 = V(1) = 5 + C_1 \Rightarrow C_1 = -4$$

$$\Rightarrow | \underline{V(t) = 4t^3 + t^2 - 4} |, \text{ thus}$$

$$P(t) = t^4 + \frac{1}{3}t^3 - 4t + C_2$$

$$0 = P(1) = 1 + \frac{1}{3} - 4 + C_2 \Rightarrow C_2 = 8/3$$

thus,

$$| \underline{P(t) = t^4 + \frac{1}{3}t^3 - 4t + 8/3} |$$

4. (15 points) a) Find the absolute extrema of  $f(x) = x^{2/3}(x - 6)$  on the interval  $[-1, 5]$ .

$$f'(x) = \frac{5}{3}x^{2/3} - 4x^{-1/3} = x^{-1/3} \left( \frac{5x}{3} - 4 \right) \Rightarrow \text{C.P.s } \textcircled{1} x=0 \text{ and } \textcircled{2} x = 12/5$$

$$f(-1) = -7 \quad \longleftarrow \text{abs min}$$

$$f(12/5) \approx -6.45$$

$$f(0) = 0 \quad \longleftarrow \text{abs. max}$$

$$f(5) = -\sqrt[3]{25} \quad \cancel{\text{at } x=5 \text{ abs max}}$$

- b) Find the absolute extrema of  $f(x) = (x - 3)^{2/3}$  on the interval  $[2, 11]$ .

$$f'(x) = \frac{2}{3}(x-3)^{-1/3}, \quad \text{only C.P. } \textcircled{1} x=3.$$

$$f(2) = 1$$

$$f(3) = 0 \quad \longleftarrow \text{abs min}$$

$$f(11) = (\sqrt[3]{8})^2 = 4 \quad \longleftarrow \text{abs. max}$$

- c) Find the absolute extrema of  $f(x) = \frac{x^3}{3} - 2x^2 + 3x$  on the interval  $[0, 4]$ .

$$f'(x) = x^2 - 4x + 3 = (x-1)(x-3) \quad \text{C.P.s } \textcircled{1} x=1, \textcircled{2} x=3$$

$$f(0) = 0 \quad \longleftarrow \text{abs min}$$

$$f(1) = \frac{4}{3} \quad \longleftarrow \text{abs max}$$

$$f(3) = 9 - 18 + 9 = 0 \quad \longleftarrow \text{abs min}$$

$$f(4) = \frac{64}{3} - 32 + 12 = \frac{4}{3} \quad \longleftarrow \text{abs max}$$

5. (20 points) Consider the function  $f(x) = \frac{x^2 - 1}{x^2 + 1}$ .

a) Determine the interval(s) where  $f(x)$  is positive/negative.

$$f(x) > 0 \quad \text{on} \quad (-\infty, -1) \cup (1, \infty)$$

$$f(x) < 0 \quad \text{on} \quad (-1, 1)$$

b) Find  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$ . Give the equations of any asymptotes.

$$\text{Wavy Line} \quad \lim_{x \rightarrow \pm\infty} f(x) = 1$$

$$\text{H.A. } @ y = 1$$

c) Give the intervals of increase and decrease and give the coordinates of any local min/max (meaning the  $x$  and  $y$  values).

$$f'(x) = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

$$f' \leftarrow \begin{array}{c} - \\ \text{---} \\ + \end{array} \quad \text{at } x=0$$

$f$  is inc. on  $[0, \infty)$  and dec. on  $(-\infty, 0]$ .

$f$  also has a local min  $\circledast$  at  $x=0$  ( $P=(0, -1)$ ) by the F.D.T.

d) Find the intervals of concavity and the coordinates of any inflection points.

$$f''(x) = \frac{4(x^2+1)^2 - 2(x^2+1) \cdot 2x \cdot 4x}{(x^2+1)^4} \quad f \text{ is C.U. on } \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{4(x^2+1) - 16x^2}{(x^2+1)^3}$$

and C.D. on

$$(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$$

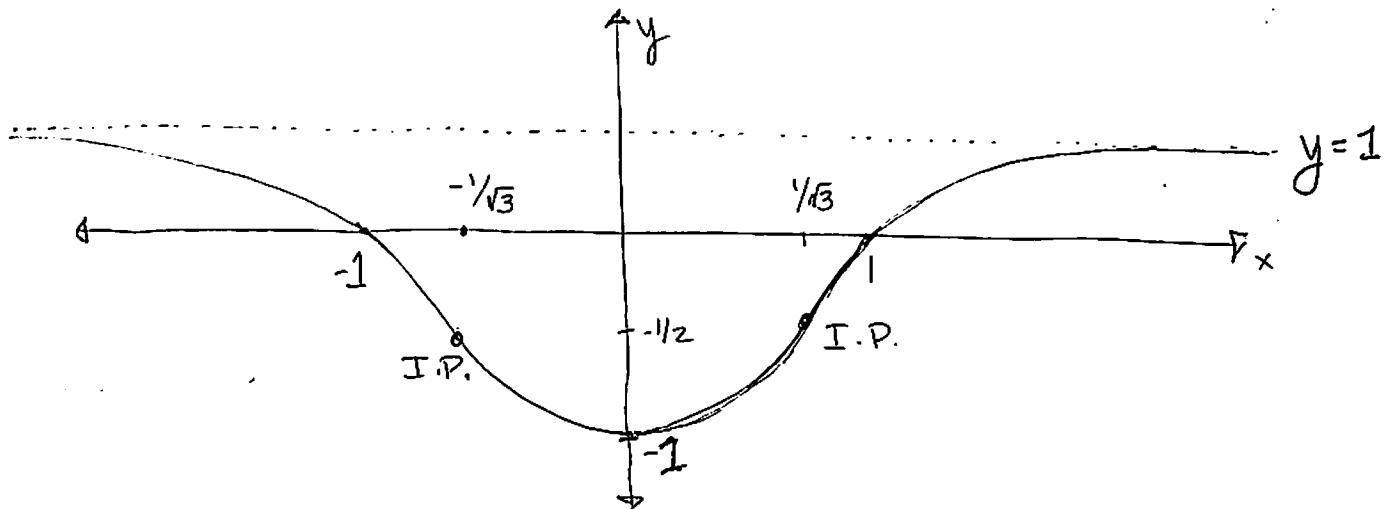
and therefore  $f$  has I.P.s

$$\textcircled{a} \quad x = \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cdot \underline{\text{AKA}} \textcircled{a}$$

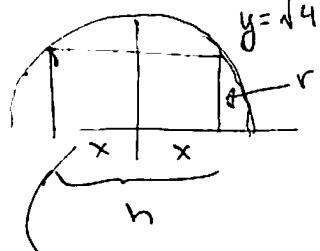
$$\text{I.P.}_1 = \left(\frac{-1}{\sqrt{3}}, \frac{-1}{2}\right)$$

$$\text{I.P.}_2 = \left(\frac{1}{\sqrt{3}}, \frac{-1}{2}\right)$$

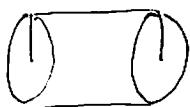
6. (10 points) Sketch a graph of the function from the previous page. Label the asymptotes, extrema and inflection points.



7. (20 points) a) Find the radius and height largest right circular cylinder that can fit inside a sphere of radius 2.



$$\begin{aligned} V_{\text{cyl}} &= \pi r^2 h \\ &= \pi (\sqrt{4-x^2})^2 2x \quad \leftarrow D: [0, 2] \\ &= \pi (4-x^2) 2x \\ &= \pi (8x - 2x^3) \end{aligned}$$

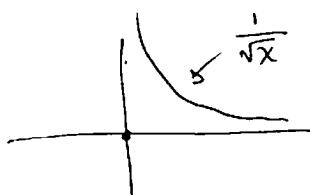


$$V'_{\text{cyl}} = \pi (8 - 6x^2) \Rightarrow x = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \text{ is max}$$

because  $V_{\text{cyl}}(0) = V_{\text{cyl}}(2) = 0$ .

Largest  $\square$   $h: \frac{4}{\sqrt{3}}$ ,  $r: \sqrt{4 - (\frac{4}{\sqrt{3}})^2} = \sqrt{4 - \frac{16}{3}} = \sqrt{\frac{4}{3}}$  ~~and then~~

- b) What point on the graph of  $f(x) = \frac{1}{\sqrt{x}}$  is closest to the origin?



$$d = \sqrt{(\frac{1}{\sqrt{x}} - 0)^2 + (x - 0)^2} = \sqrt{\frac{1}{x} + x^2}$$

$$d' = \frac{1}{2\sqrt{\frac{1}{x} + x^2}} \left( \frac{-1}{x^2} + 2x \right)$$

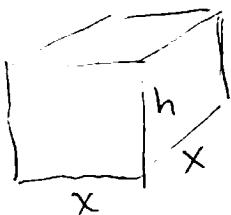
$\left[ \begin{array}{l} \text{so closest pt. is} \\ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \end{array} \right]$

So, if  $2x - \frac{1}{x^2} = 0$

$$\frac{2x^3 - 1}{x^2} = 0 \quad \text{and thus, } x = \frac{1}{\sqrt[3]{2}}.$$

$d'$   $\leftarrow - \rightarrow +$   $\leftarrow$  Proves it's a min.  $\leftarrow$  (it's the only C.P.)

- c) You are to design a, quite large, square bottom box with total volume of 1500 ft.<sup>3</sup>. The mysterious material you are to use costs 2 dollars per ft.<sup>2</sup> and you need to use two sheets of mysterious material on the bottom (this makes the box stronger). Find the dimensions and cost of the cheapest box you can make.



Given:  $h x^2 = 1500 \rightarrow h = \frac{1500}{x^2}$

$$\begin{aligned} C &= 2 \cdot (3x^2) + 2(4xh) = 6x^2 + 8xh \\ &= 6x^2 + 8x \left( \frac{1500}{x^2} \right) \end{aligned}$$

$$= 6x^2 + \underbrace{\frac{12000}{x}}_D: (0, \infty)$$

$$C'(x) = 12x - \frac{12000}{x^2}$$

$$= \frac{12x^3 - 12000}{x^2} \Rightarrow x = 10$$

$C'$   $\leftarrow - \rightarrow +$

so, it's a min. and the dim's are  $10 \times 10 \times 15$